Non-Planar Knot Graphs

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1 Introduction

Knot graphs or knot shadows are planar two-dimensional representations of knot diagrams [1]. By use of the "Splitting Rule" to generate new knot graphs, planar and non-planar graphs my be generated. In [1] the problem to decide wether a graph is planar or not, was solved intuitively just by looking at it and trying to redraw it into planar form.

This document presents a first try of a formal solution of the problem.

We will use the terminology of [1], therefore speaking of Crossings c instead of vertices and of Strands s in lieu of edges. Furthermore we will count the inner Areas a, enclosed by strands.

2 Properties of Planar Knot Graphs

First, let us analyze known planar knot graphs.

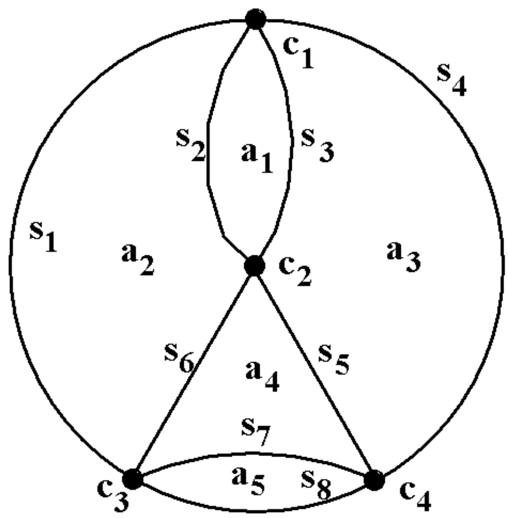

Figure 1: Quantitative Properties of the Figure Eight Knot.

The knot graph of the Figure Eight Knot K4.1 consists of 4 crossings, 8 strands and 5 enclosed areas.

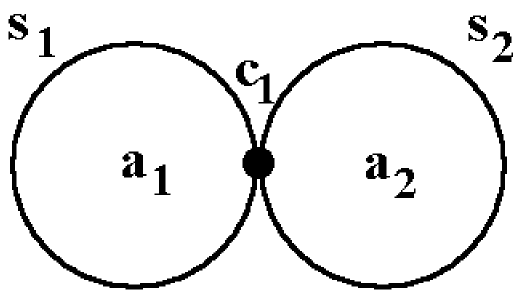

Figure 2: Quantitative Properties of the Turned Knot.

The knot graph of the Turned Knot K1.1, that is equivalent to the Unknot, consists of 1 crossing, 2 strands and 2 enclosed areas.

2.1 Crossings and Strands

Every Crossing is generated by two crossing Strands. Therefore, in the knot graph four strands depart every crossing. Because knot graphs are closed each strand begins and ends at a crossing. If C is the number of crossings so $4 \cdot C$ is the number S of strands departing the crossings. But while each strand ends at a crossing the number of strands is $\frac{4 \cdot C}{2}$. So

$$S = 2 \cdot C \tag{1}$$

2.2 Areas and Crossings

The proposition is that the number A of enclosed areas is C+1. We will prove this by induction.

Begin of the induction is e.g. the knot graph K1.1 of Figure 2, which has one crossing, two strands, and two enclosed areas. Therefore, the proposition is true in this case.

Induction: If a knot graph has i crossings $c_1, .., c_i$ and, according to equation (1) $2 \cdot i$ strands, it has j enclosed areas $a_1, .., a_j$. Here the number of crossings is $C(c_i) = C = i$ and the number of areas is $A(a_j) = A = j = C + 1$

according to the induction proposition.

When an additional crossing c_{i+1} is introduced it is inserted into an existing strand. To fulfill the condition of four strands connected to the new crossing, a strand in the vicinity is to be split and connected. If, without loss of generality, the strand in which the crossing was inserted is a border strand of the knot graph, the two strands bordered two areas before the operation was carried out. After the split, the former two areas still exist, but a new area a_{j+1} was created.

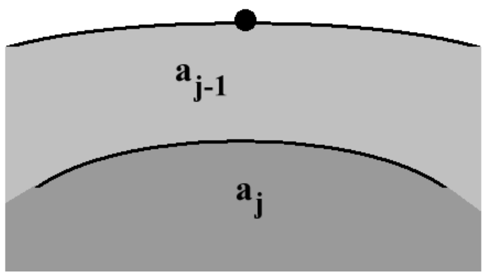

Figure 3: New cossing.

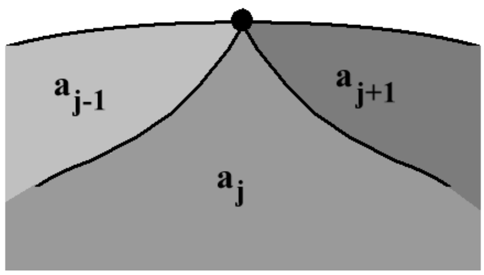

Figure 4: New cossing and new area.

The new crossing c_{i+1} increased the number of crossings by one, $C(c_{i+1}) = i + 1 = C + 1$, and created an additional area a_{j+1}. So, the number of areas $A(A_{j+1}) = j+1 = (C+1)+1 = A+1$, applying the induction proposition.

From $(C + 1) + 1 = A + 1$ we get

$$A = C + 1 \tag{2}$$

q.e.d.

2.3 Knot Graph Equation

If we add the equations (1) and (2) we get

$$S + A - 3 \cdot C - 1 = 0 \tag{3}$$

what we will call the **Knot Graph Equation**.

Note: *This Knot Graph Equation is true even if the knot graph represents no knot but a link! And, it is equivalent to Euler's polyeder formula without the external area.*

3 Non Planar Graphs

Applying the Splitting Rule to existing planar knot graphs there may graphs be generated with strands bridging each other in the third dimension. Some of these graphs are planar, others are not.

First, a splitted graph is shown that turns out to be planar. The transformation is made intuitively.

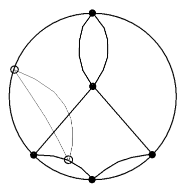

Figure 5: Splitted knot graph, initial form.

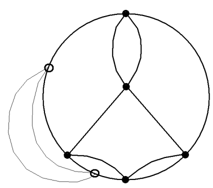

Figure 6: Splitted knot graph, planar form.

Others are in some case non-planar (still intuitively).

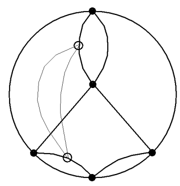

Figure 7: Another splitted knot graph, initial form.

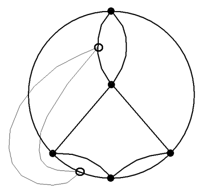

Figure 8: Transformed splitted knot graph, still non-planar.

3.1 Conjectures

The Knot Graph Equation in the present form does not provide much help to the distinction of planar and non-planar graphs.

Conjectures like the followings turn out to be false by counterexamples:

If $S + A - 3 \cdot C - 1 \neq 0$ is odd then the graph is non-planar and therefore no knot graph.

If $S + A - 3 \cdot C - 1 \neq 0$ is even then the graph is planar and represents a knot graph.

Helpful properties are still to be discovered.

4 Sources

References

[1] Wolf Scheuermann:
Knots and Links v3.0,
Structures and Classifications.
www.Forschungskontor.de 2015